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A Cluster of Integer Solutions to Homogeneous Ternary Quadratic Equation

$$x^2 = y^2 + z^2 + xy + yz + zx$$

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Abstract

The homogeneous quadratic Diophantine equation with three variables given by $x^2 = y^2 + z^2 + xy + yz + zx$ is considered to get different patterns of integer solutions .Substitution technique is employed and solutions of Pythagorean equation are utilized to obtain the required integer solutions.

Keywords: Homogeneous quadratic , Ternary quadratic ,Substitution technique, Integer solutions

Notations

$$t_{3,n} = \frac{n(n+1)}{2}$$

$$P_n^3 = \frac{n(n+1)(n+2)}{6}$$

$$P_n^5 = \frac{n^2(n+1)}{2}$$

$$O(n) = \frac{n(2n^2+1)}{3}$$

$$SO(n) = n(2n^2-1)$$

$$S(n) = 6n(n-1)+1$$

Introduction

The subject of quadratic Diophantine equations has a rich variety of fascinating problems. The homogeneous or non-homogeneous quadratic equations with three unknowns are rich in variety and attracted many mathematicians. For an extensive review of sizable literature and various problems, one may refer [1-8]. This paper focuses on finding various choices of non-zero distinct integer solutions to the homogeneous ternary quadratic Diophantine equation $x^2 = y^2 + z^2 + xy + yz + zx$. Different sets of integer solutions are obtained by employing the linear transformations and solutions of Pythagorean equation. Some interesting relations among the integer solutions are presented.

Method of analysis

The homogeneous ternary quadratic equation to be solved is

$$x^2 = y^2 + z^2 + xy + yz + zx \quad (1)$$

On scrutiny, (1) is satisfied by

$$(x, y, z) = (14,10,-2), (-6,10,-2), (-7,1,11), (-7,-5,11)$$

However, there are many more integer solutions to (1) that are illustrated below:

Illustration 1

The option

$$y = u + v, z = u - v, u \neq \pm v \neq 0 \quad (2)$$

in (1) leads to the second degree equation in x given by

$$x^2 - 2ux - (3u^2 + v^2) = 0$$

On solving the above equation for x ,one obtains

$$x = u \pm \sqrt{4u^2 + v^2} \quad (3)$$

The square-root on the R.H.S. of (3) is eliminated when

$$u = rs\ , v = r^2 - s^2,\ r > s > 0 \quad (4)$$

Thus,from (2) &(3) ,the following two sets of integer solutions to (1) are obtained:

Set 1 : $x = rs + r^2 + s^2, y = rs + r^2 - s^2, z = rs - r^2 + s^2$

Set 2 : $x = rs - r^2 - s^2, y = rs + r^2 - s^2, z = rs - r^2 + s^2$

A few interesting relations among the integer solutions to (1) in Set 1 are presented below:

(i) $(x - y)(x - z) = (y + z)^2$

(ii) $4x - 2(y + z)$ is written as sum of two squares

(iii) $2x-(y+z)$ is a square multiple of 2 when r & s are taken to represent legs of Pythagorean triangle

(iv) When $r=a(a^2+b^2), s=b(a^2+b^2)$,it is seen that $2x-(y+z)$ is twice a cubical integer

(v) When $r=a(a^2+b^2), s=b(a^2+b^2)$,it is seen that $8x-4(y+z)$ is a cubical integer

(vi) $(2x-y-z)^2=(y-z)^2+4(y+z)^2$

(vii) $6x-3(y+z)$ is a square multiple of 6 when r & s are taken to represent legs of Pythagorean triangle

(viii) $r=t_{3,s+1} \Rightarrow y+z=6P_s^3$

(ix) $r=s(s+1) \Rightarrow y+z=4P_s^5$

(x) $r=(2s^2+1) \Rightarrow y+z=6O(s)$

(xi) $r=(2s^2-1) \Rightarrow y+z=2SO(s)$

(xii) $r=3(s-1) \Rightarrow y+z+1=S(s)$,

Note 1

Apart from (4) , the square-root on the R.H.S. of (3) is eliminated when

$$u=2(r^2-s^2), v=8rs \quad (5)$$

For this choice , the corresponding two sets of integer solutions to (1) are as presented below:

Set 3 : $x=6r^2+2s^2, y=2(r^2-s^2)+8rs, z=2(r^2-s^2)-8rs$

Set 4 : $x=-2r^2-6s^2, y=2(r^2-s^2)+8rs, z=2(r^2-s^2)-8rs$

Illustration 2

The option

$$z=u+v, x=u-v, u \neq \pm v \neq 0 \quad (6)$$

in (1) leads to the second degree equation in y given by

$$y^2+2uy+4uv+u^2-v^2=0$$

On solving the above equation for y ,one obtains

$$y=-u \pm \sqrt{v^2-4uv} \quad (7)$$

The square-root on the R.H.S. of (7) is eliminated when

$$u = rs\,, v = (r+s)^2 \qquad (8)$$

Thus,from (6) &(7) ,the following two sets of integer solutions to (1) are obtained:

Set 5 : $x = -(rs + r^2 + s^2), y = -rs + r^2 - s^2, z = 3rs + r^2 + s^2$

Set 6 : $x = -(rs + r^2 + s^2), y = -rs - r^2 + s^2, z = 3rs + r^2 + s^2$

Note 2

Apart from (8) , the square-root on the R.H.S. of (7) is eliminated when

$$u = 2(r^2 - s^2), v = 8r^2, r > s > 0 \qquad (9)$$

For this choice , the corresponding two sets of integer solutions to (1) are as presented below:

Set 7 : $x = -6r^2 - 2s^2, y = -2(r^2 - s^2) + 8rs, z = 10r^2 - 2s^2$

Set 8 : $x = -6r^2 - 2s^2, y = -2(r^2 - s^2) - 8rs, z = 10r^2 - 2s^2$

As in illustration 1,similar relations among the solutions may be found.

Conclusion

An attempt has been made to obtain various sets of integer solutions to the homogeneous second degree equation with three unknowns given by $x^2 = y^2 + z^2 + xy + yz + zx$. As the ternary quadratic equations are plenty, the readers and researchers may search for other choices of ternary quadratic equations to determine their integer solutions.

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